→ Transport

- previously developed theory of fluctuations near equilibrium
- now, → transport / relaxation
- prblm. is one of Coulomb scattering in stable plasma (i.e. resistivity)

⇒ weak deflection $\qquad \Delta p_\perp \sim \int_{-\infty}^{+\infty} dt \frac{e^2}{r^2} \frac{\rho}{r}$

- Approach via:
  - general theory of pdf evolution via series of small, random kicks.

    ⇒ Diffusion, Central Limit Thm.
  - Fokker-Planck Eqn. (Generic)
  - Lenard-Balescu Eqn. (extends test particle model)
  - FPE + Rosenbluth Potentials

→ Basics:

Can discretize process (N jumps) with probability of jump by $\underline{r}$ in step.

$\underline{o}$

1 step back

(no memory)

$$P_{N+1}(\underline{X}) = \int p(\underline{r})\, P_N(\underline{X}-\underline{r})\, d\underline{r}$$

step pdf

$\underline{X}$ = position, cumulative

$\underline{r}$ = jump

For small steps, expand to 2nd order:

$$P_{N+1}(\underline{X}) = \int d\underline{r} \Big\{ P_N(\underline{X}) - \underline{r}\cdot\underline{\nabla} P_N(\underline{X}) + \tfrac{1}{2}\, \underline{r}\cdot\underline{\nabla}\,\underline{\nabla} P_N\cdot\underline{r} \Big\}\, p(\underline{r})\, d^d\underline{r}$$

$$\simeq P_N(\underline{X}) + \frac{\langle \underline{r}\cdot\underline{r}\rangle}{2d} \nabla^2 P_N(\underline{X})$$

no bias

→ normalizable p, $\langle r^2 p\rangle$ exists.

→ no bias

→ thus, have:

$$\frac{P_{N+1}(\underline{X}) - P_N(\underline{X})}{\Delta t} = \frac{\langle r^2\rangle}{2d\,\Delta t} \nabla^2 P_N$$

diffusion equation!

$$\boxed{\frac{\partial P}{\partial t} = D \nabla^2 P \qquad D = \langle r^2\rangle / 2d\,\Delta t}$$

$p \to \rho$

$$\rho(\underline{x}, 0) = \delta(\underline{x})$$

so

$$\frac{\partial \rho}{\partial t} = -Dk^2 \rho$$

FT

$$\rho(k, t) = e^{-Dk^2 t} \rho(k, 0)$$
$$= e^{-Dk^2 t} \quad 1.$$

so Inverse $\Rightarrow$

$$\boxed{\rho(\underline{x}, t) = \frac{e^{-x^2/4Dt}}{(4\pi D t)^{d/2}}} \to$$

pdf for position, in time

i.e.

$$\rho(\underline{k}, t) = e^{-Dk^2 t}$$
$$= e^{-D(k_x^2 + k_y^2 + k_z^2)t}$$

so

$$\rho(\underline{x}, t) = \int e^{i(k_1 x_1 + k_2 x_2 + k_3 x_3 + \cdots)} e^{-D(k_1^2 + k_2^2 + \cdots k_d^2)t} \, dk_1 \, dk_2 \cdots dk_d$$

$$= \text{above.}$$

$\therefore$

$$\rho(\underline{x}, t) \sim \frac{1}{t^{-d/2}} e^{-x^2/4Dt}$$

$$x^2 = \underline{x} \cdot \underline{x}$$

Then, re-discretizing:

$$P_N(\underline{x}) = \frac{\exp[-dx^2/2\langle r^2\rangle N]}{(2\pi\langle r^2\rangle N/d)^{d/2}}$$

$$\boxed{P_N(x) \sim \frac{e^{-d\,x^2/2\langle r^2\rangle N}}{(\langle r^2\rangle N)^{d/2}}}$$

$P_N(x)$:

- converges, times asymptotically, to Gaussian, with width $\sim N^{1/2}\langle r^2\rangle^{1/2} \sim N^{1/2} r_{rms}$
- $\sim N^{-d/2}$.

$\Rightarrow$ Quickie demo of Central Limit Theorem.

C.L.T.:

Let $X_1, X_2, \ldots$ be a sequence of:

- independent
- identically distributed

random variables each with mean μ and variance σ^2

i.e. $\overline{X_i} = \mu$

$$\langle (X_i - \overline{X_i})^2 \rangle = \sigma^2 .$$

Then the distribution of:

$$\frac{X_1 + X_2 + \cdots X_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{n \to \infty} \text{normal (Gaussian)}$$

i.e.

$$P\left\{ \frac{X_1 + X_2 + \cdots X_n - n\mu}{\sigma\sqrt{n}} \leq \varepsilon \right\}$$

$$= \int_{-\infty}^{\varepsilon} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx$$

Note: Holds for all sequences

- independent $\Rightarrow$ no correlations
- identically distributed
  i.e. no 'special' step, $\Rightarrow$ intermittency.
- σ^2 exists $\Rightarrow$ no fat tails

Related: Law of Large #'s

Let $X_1, X_2 \ldots$ be a sequence of random variables having a common distribution and let

$$E(X_i) = \mu$$

expectation

Then, with probability 1:

$$\frac{X_1 + X_2 + \ldots X_n}{n} \rightarrow \mu$$

as $n \rightarrow \infty$

In simple terms:

$n \to \infty$, X_i random variable

- LLN: Average conv. to $E(X_i)$
- CLT: Sum Distribution $\to$ Gaussian width $N\sigma^2$, centroid $N\mu$.